THE OPINION GAME: STOCK PRICE EVOLUTION FROM MICROSCOPIC MARKET MODELLING

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Abstract. We propose a class of Markovian agent based models for the time evolution of a share price in an interactive market. The models rely on a microscopic description of a market of buyers and sellers who change their opinion about the stock value in a stochastic way. The actual price is determined in realistic way by matching (clearing) offers until no further transactions can be performed. Some analytic results for a non-interacting model are presented. We also propose basic interaction mechanisms and show in simulations that these already reproduce certain particular features of prices in real stock markets.

1. Introduction

The financial markets constitute an intriguing and complex system that has not failed to attract mathematicians and scientists from other fields for a long time. Only rather recently, however, has mathematical finance, and more specifically the theory of derivatives on the stock markets become a major field of mathematics and one of the major sources of inspiration for probability theory in general and stochastic analysis in particular. The reason for this development is simple: It is based on the apparent success of the so-called Black-Scholes formula for the fair price of an option as a tool for the actual trader on the market. Indeed, the very existence of this mathematical theory appears to be largely responsible for the recent growth and diversification of the derivative market itself, which in its present form would have been impossible without an underlying mathematical theory. On the other side, the great success of this same theory in the mathematical sciences is due, beyond the obvious advantages it provides for the careers of students trained in this field, largely to the fact that there is a very clear mathematical setting for this theory with clearly spelled out axioms and assumptions which allows for the mathematician to bring his traditional weapons to bear in a familiar terrain.

One of the crucial issues in the financial mathematics is the modeling of prices of commodities (stocks, currencies *etc.*) with help of stochastic processes. The main approaches that have been used in this context are the following:

- *generalized Black-Scholes (BS) theory.* Originally the BS theory [2] (for good textbook exposition see, *e.g.*, [10, 7]) emerged as a theoretical foundation of pricing of derivatives (options) of underlying financial instruments (stocks, currencies *etc.*). Initially the price of the underlying was taken as geometric Brownian motion. Later this theory was generalized and put on

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the axiomatic background based on the assumption of the so-called non-arbitrage condition. This led to the conclusion that prices are described by semi-martingale measures, and thus are essentially given by solutions of certain stochastic differential equations. This framework has been the main driving force of the rapid growth of financial mathematics in the last decades. Note that the main purpose of this theory is not to derive specific models for the underlying but rather to deduce consequences from generally accepted principles. In this respect this approach can be compared to classical thermodynamics.

- *statistical approach.* Empirical studies of share price data try to model the data by certain stochastic process. Popular classes of models are ARCH, GARCH, ARMA *etc.* There seem to be no totally conclusive results, but certain interesting phenomena have been observed, such as universal exponents in certain correlation data (see [1]). Quite frequently, analogies to phenomena like turbulence are drawn. This approach does not usually intend to derive the model from any underlying economic theory.
- *agent based models.* "As prices are generated by the demand of agents who are active on the financial market for the given asset, a [...] model [...] should be explained in terms of the interaction of these agents." Based on this observation stated by Föllmer in [6], a large number of agent based models for price evolution have been developed. Föllmer in [6] suggested a model of diffusion in random environment by viewing the price process as a sequence of temporary equilibria in a market with agents. Among the most popular microscopic models for financial markets are so-called agent based models that are more or less sophisticated versions of the "minority game" (MG); for reviews, see Jeffries and Johnson [9], Bouchaud and Giardina [8], and references therein. In all these models there exists a collection of traders, each endowed with the possibility to make a decision (typically of the type "buy", "sell", or "hold") concerning a given investment. To reach such a decision, the trader disposes of a certain number of "strategies", a strategy typically being a function of the price history into the set of decisions. The game consists of the traders choosing their strategies in a way as to reach certain objectives (in the minority game to be "in the minority", in reality to make a maximal profit from the transactions undertaken). At each time step, the asset price is updated according to an empirical rule as a function of the number of "buyers" and "sellers". There are currently a large number of versions of such models around, including models with additional stochastic components. These models exhibit a rather rich dynamical structure. However, they are rather heavy handed both analytically (where little or nothing is known on a mathematical level) and numerically. Moreover, the large number of assumptions and parameters entering the models makes their predictive power somewhat limited. Purely deterministic models of this type have been criticized before (Bouchaud and Giardina [8]) and stochastic models have been proposed that should allow to take into account irrational behaviour of agents.

In this paper we want to propose agent based market models that are *much simpler* and that, at the same time, are build reasonably close to what is actually happening in financial markets. The main distinction between our model and the

MG type models is that it focuses on collective effects of a market while not attempting to model the actual reasoning process of an individual trader. The basic paradigm of our modeling approach is the notion of a price. Prices of a share of stock or other commodity arise from trading. There are various developed theories in economics concerning prices based on some equilibrium assumptions, but, fortunately, in the stock market in particular, the price of a share is obtained by a well defined procedure which is easily implementable in an algorithm provided a sufficient amount of information about the state of the market at any given time is available. The basic principle of our approach is the modeling of a set of interacting agents in a way that allows to extract the price from the current trading state in a rigorous way.

The remainder of this article is organized as follows. In Section 2 we explain the basic principles of our modeling approach in detail and illustrate them in some special cases. In Section 3 we analyze some of the phenomena that can already be observed in this simplest setup and relate them to certain features of real world data. In Section 4 we introduce a number of additional features that should be implemented to obtain more realistic models, and discuss some of the additional effects they produce. In Section 5 we present our conclusions.

2. Our approach

The basic procedure of price determination in real stock markets is done by trading. That is to say, participating traders propose prices at which they are willing to sell respectively buy a certain number of shares of a given stock. The list of these offers at a stock exchange is called the orderbook. On the basis of this orderbook the market maker is matching buy and sell offers according to certain rules until no further transactions are possible. That is to say, after the matching procedure (clearing) the highest price proposed for buying a share is smaller than the lowest price at which a share is offered. These two values are then quoted as the bid price and the ask price. The dynamics of the real stock market thus has two components:

- the changing of buy and sell offers made by traders and
- the matching of these offers by the market maker which fixes the quoted price at a given time.

Our purpose is to develop a class of models which reflects this mechanism of pricing and allows for diverse modeling on different levels of complexity of the behaviour of the agents, while maintaining the pricing mechanism by the market maker. To do so, a minimal requirement for the description of the state space of the trading agents is that it must allow us to recover the state of the orderbook at any given moment in time.

Our idea is thus to consider the time evolution of a virtual orderbook or a "trading state" containing the opinions of each participating agent about the "value"[1] of the stock. The evolution is driven by the change in opinion of the agents and the action of the market maker.

[1]We will distinguish the notion of the value from that of the price. The *value* is what agents have an opinion about, while the *price* is determined by the market. The opinion on the value can be driven by fundamental considerations (*e.g.* earning or dividend expectations, typically coming from outside information), or speculative considerations (*e.g.* predictions based on partial knowledge on the current state of the opinions of other traders), or both.

A minimal model in which this idea can be implemented can be described as follows. We consider trading in one particular stock. Assume that there are N "traders" and $M < N$ shares of the stock. We make the simplifying assumption that each trader can own at most one share. The state of *each* trader i is given by its opinion $p_i \in \mathbb{R}$ about the logarithm of the value of the stock, and by the number of shares he owns $n_i \in \{0, 1\}$. This is to say, the trader i would be willing to sell his share at the price e^{p_i}, if he owns one ($n_i = 1$), respectively buy a share at this price, if he does not own one ($n_i = 0$). We say that a trading state is *stable*, if the M traders having the M highest opinions p_i all own a share. This means in particular that in a stable state one can infer the set of owners of shares from the knowledge of the state of opinions $\mathbf{p} = (p_1, \dots, p_N)$. Thus a stable trading state is completely determined by the set of N values p_i, and we will in the sequel identify stable trading states with the vector $\mathbf{p}$. As we will normally only work with stable trading state, we suppress the qualifier stable when no confusion can arise.

Given a stable trading state $\mathbf{p}$ we denote by $\hat{\mathbf{p}} = (\hat{p}_1, \dots, \hat{p}_N)$ its order statistics, that is $\hat{p}_i = p_{\pi_i}$ for a permutation $\pi \equiv \pi(\mathbf{p})$ of the set of N elements such that $\hat{p}_1 \le \dots \le \hat{p}_N$. Then the number of shares owned by traders, $\mathbf{n}(\mathbf{p}) = \big(n_1(\mathbf{p}), \dots, n_N(\mathbf{p})\big)$ satisfies

$$n_i(\mathbf{p}) = \begin{cases} 0, & \text{if } \pi_i(\mathbf{p}) \le N - M, \\ 1, & \text{if } \pi_i(\mathbf{p}) \ge N - M + 1. \end{cases}$$

To a trading state $\mathbf{p}$ we associate the *ask price*

$$p^a \equiv p^a(\mathbf{p}) = \hat{p}_{N-M+1}\,,$$

and the *bid price*

$$p^b \equiv p^b(\mathbf{p}) = \hat{p}_{N-M}\,.$$

Obviously, $p^a(\mathbf{p})$ is the lowest price asked by traders owning a share and $p^b(\mathbf{p})$ is the highest price offered by traders wanting to buy a share. For convenience we will refer to $\frac{1}{2}(p^a + p^b)$ as the *current price* in the sequel.

Any dynamics $\mathbf{p}(t)$ defined on the trading state induces the dynamics of $\hat{\mathbf{p}}$ and in particular of the pair $\big(p^a, p^b\big)$.

Our next simplifying assumption is that the above trading state $\mathbf{p}$ evolves in time as a (usually time-inhomogeneous) Markov chain[2] $\mathbf{p}(t)$ with state space $\mathbb{R}^N$. We will further assume that time is discrete (this is inessential but more convenient for computer simulations) and that the updating proceeds asynchronously, *i.e.* typically at a given instant only a single opinion changes. This dynamics can be considered as an interacting particle system, however, some special features should be incorporated that reflect the peculiarities of a market.

The first and most obvious one is that the ask and bid prices $p^a(t), p^b(t)$ are likely to be important quantities which will influence the updating probabilities.

Moreover, it is reasonable to distinguish between transitions that leave $\mathbf{n}$ unchanged, and those for which $\mathbf{n}\big(\mathbf{p}(t+1)\big) \neq \mathbf{n}\big(\mathbf{p}(t)\big)$. In the latter case we say that a *transaction* has occurred.

Before we discuss some more specific implementations of this general setup a few remarks concerning some features that may appear offending are in place.

The first is doubtlessly the assumption that the process is Markovian. This appear unnatural because the most commonly available information on a stock is the history of its price, the "chart", and most serious traders will take this

[2]We will discuss the Markovian assumption later.

information at least partly into account when evaluating a stock, with some making it the main basis for any decision.[3] Certainly one could retain such information and formulate a non-Markovian model, as *e.g.* the model of Bouchaud and Giardina [8]. However, if one starts to think about this, one soon finds that it is very difficult to formulate reasonable transition rules on the basis of the price history. On a more fundamental level, one will also come to the conclusion that the analysis of the history of the share price is in fact performed in order to obtain information of the *current opinion of the traders concerning the value of the stock* with the hope of inferring information on the future development of the price. For instance, if one knew that there are many people willing to sell shares at a price not much higher than p^a, one knows that it will be difficult for the price to break through this level (this is know as a "resistance" by chart-analysts and usually inferred from past failures to break through such a level). Therefore, instead of devising rules based on past price history, we may simply assume that the market participants have some access to the prevailing current opinions, obtained through various sources (chart analysis, rumours, newspaper articles, *etc.*) and take this into account when changing their own estimates.

The second irritating point is that money does not appear in our model except in the form of opinions about values. In particular, we do not keep track of the cashflow of a given investor (that is to say we do not care whether a given investor wins or looses money). There are various reasons to justify this. First we consider that the market participants do not invest a substantial fraction of their assets in this one stock, so that shortage of cash will not prevent anyone to buy if she deems opportune to do so (in the worst case money can be obtained through credits). Then, money is not conserved, but the total value of the stock can inflate as long as there is enough confidence. Also, we do not keep track of the objective success of a trader, because we do not know how this will eventually influence her decisions. While a given trader may follow her personal strategy with the hope of making profits, we cannot be sure that these strategies will succeed. What is important and what is built into our model, however, is the fact that any trader[4] will have the subjective impression to make a profit at any transaction.[5] Thus we feel that opinions about values are the correct variables to describe such a market rather than the actual flow of capital, at least at the level of a simple model.

The above setting suggests a rather general and flexible class of models of a stock market. Its main feature is that it describes the time evolution of a share price as the result of an interacting random process that reflects the change of the opinions of individual traders concerning the value of the stock. Even when this last process is modelled as a Markov process, the resulting price process $\big(p^a(t), p^b(t)\big)$ will in general not be a Markov process.

[3] A frequently heard remark being that all information on a stock is in its price.

[4] We could enlarge the model to incorporate a small fraction of traders which do not act according to common sense, or against their own convictions (*e.g.* traders that have bought their stock on credit and are executed by their creditors on falling prices. It may be interesting to consider the effect of that in the context of market crashes).

[5] Which also implies that this subjective opinion must be wrong at least for one of the traders involved. But this seems to reflect reality.

3. Examples

3.1. **Ideal gas approximation.** Obviously the simplest model for the dynamics of the trading state $\mathbf{p}(t)$ is to choose it as a collection of independent identically distributed one-dimensional Markov processes ("random walks") $p_i(t)$. This corresponds to the ideal gas approximation in statistical mechanics. In this case, the price process is simply obtained from the order statistics of independent processes and asymptotic results for M and/or N large (recall that M denotes the number of traded shares and N the total number of traders) can be obtained rather easily. While this model is somewhat simplistic, some rather interesting phenomena can already be modelled in this context, as we will explain now.

We may be interested in a situation where some macro-economic model may predict several stable (respectively metastable) values of the stock price, realized as the minimum of some utility function V.

In such a situation it seems not unreasonable to model the process of a single trader as a one-dimensional diffusion process with drift obtained from a potential function V, *i.e.* we can take $p_i(t)$ to be a solution of the stochastic differential equation

$$dp_i(t) = -V'\big(p_i(t)\big)dt + \sqrt{\varepsilon}\, dW_i(t)$$

with $W_i(t)$ i.i.d. standard Brownian motions, and $\varepsilon > 0$ a parameter measuring the diffusivity. Alternatively, we can take discrete approximations of this process, as will always be done in numerical simulations.

Let us consider the situation when there are two (meta) stable values of the price, q_1 and q_2, *i.e.* the situation where the potential V has two minima (wells) at q_1 and q_2. If the potential is strong, resp. ε is small, an individual trader would typically spent long periods of time near one of the favoured values q_1 or q_2.

Let w_1 be the escape rate from the well q_1 to the well q_2 and let w_2 be the escape rate in the opposite direction. It is well known that these escape rates are exponentially small, $w_i \asymp \exp\big\{-2(V^* - V_{q_i})/\varepsilon\big\}$, where V^* is the value of the maximum of V on the interval $[q_1, q_2]$, if ε is small. Denote by A_t the number of traders in the right well q_2 at time $t \geq 0$. Suppose that A_0 is much larger than M implying that the actual price at the initial moment is situated near q_2. We are interested in describing the moment of the "crash", *i.e.* when the price moves from the right well q_2 to the left well q_1.

Then we can approximate the individual processes $p_i(t)$ by a two-state Markov chains with state space $\{q_1, q_2\}$ and transition rates w_1 and w_2. In this approximation we can compute the normalized expected number $a_t \equiv \mathbf{E}A_t/N$ of traders in state q_2 at time t as the solution to the ordinary differential equation

$$\frac{d}{dt}a_t = -w_2 a_t + w_1(1 - a_t)\,.$$

We get

$$a_t = \frac{w_1}{w_2 + w_1} + \Big(a_0 - \frac{w_1}{w_2 + w_1}\Big)\exp\big\{-(w_2 + w_1)t\big\}$$

and the crash time T_c can be defined as t such that $a_t = M/N$,

$$T_c = \frac{1}{w_2 + w_1}\log\frac{a_0(w_2 + w_1) - w_1}{\frac{M}{N}(w_2 + w_1) - w_1}$$

If the energy barrier $\Delta V \equiv V^* - V_{q_2}$ is large enough, the time for each single buyer to escape from the initial well is much larger than the relaxation time for the

system of A_t particles in the right well, and thus it is natural to expect that the system will pass through the sequence of local equilibrium states corresponding to A_t independent random walkers. Using this observation, the evolution of the price can be described in terms of the $B_t \equiv A_t - M$'s order statistic in a system of A_t random variables whose distribution is approximately Gaussian with parameters q_2 and $\bigl(V''(q_2)\bigr)^{-1}$.

To do this, let $F(\,\cdot\,)$ denote the distribution function of an individual walker conditioned to stay near q_2 and define $q_B = q(B_t)$ as the solution to the equation

$$\frac{M}{A_t} = 1 - F(q_B) \approx 1 - \Phi\bigl((q_B - q_2)\sqrt{V''(q_2)}\bigr)\,,$$

where $\Phi(\,\cdot\,)$ is the distribution function of the standard Gaussian variable. Using the well-known asymptotics for the tail distribution of $\Phi(\,\cdot\,)$,

$$\frac{u}{u^2+1}\frac{e^{-u^2/2}}{\sqrt{2\pi}} \le \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-x^2/2}\,dx \equiv 1 - \Phi(u) \le \frac{1}{u}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\,, \qquad u \to \infty,$$

we immediately get

$$q_B \approx q_2 - \Bigl(\frac{2\log(A_t/B_t)}{V''(q_2)}\Bigr)^{1/2} + \frac{\log(4\pi\log(A_t/B_t))}{\sqrt{8V''(q_2)\log(A_t/B_t)}}, \qquad B_t/M \to 0\,.$$

Consequently, in the limit of large A_t, M such that $\rho = 1 - M/A_t$ is fixed we have

$$q_B \approx q_2 - \Bigl(\frac{2\log(\rho^{-1})}{V''(q_2)}\Bigr)^{1/2}\,.$$

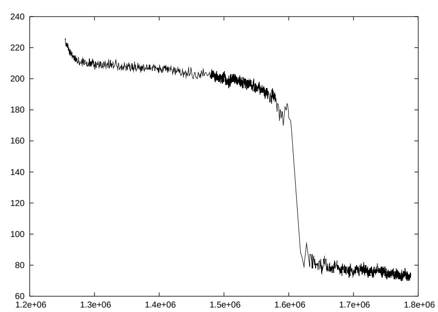

FIGURE 1. Crash in a double-well potential

In this regime of "increasing ranks", the fluctuations of the B_t's order statistic have Gaussian behaviour and their scaling can be derived from [11, Theorem 2.5.2]. To do this, consider a small enough y such that

$$A_t\,F(q_B+y)\bigl(1 - F(q_B+y)\bigr)$$

is large for large B_t and such that

$$\frac{M - A_t\bigl(1 - F(q_B+y)\bigr)}{(B_t\,M/A_t)^{1/2}} \to \tau$$

as A_t, $M = (1-\rho)A_t$, and $B_t = \rho A_t$ are getting large. In view of the definition of q_B the LHS expression above equals

$$\sqrt{\frac{B_t\,A_t}{M}}\Bigl(\frac{F(q_B+y)}{F(q_B)} - 1\Bigr)\,,$$

where the last ratio can be approximated by

$$\frac{q_2 - q_B}{q_2 - q_B - y} \frac{\exp\bigl\{-(q_B + y - q_2)^2 V''(q_2)/2\bigr\}}{\exp\bigl\{-(q_B - q_2)^2 V''(q_2)/2\bigr\}} \approx 1 + (q_2 - q_B) V''(q_2)\, y\,,$$

assuming that $(q_2 - q_B)^2 V''(q_2)$ is large enough. As a result, in the limit of large M and B_t, we have

$$\tau \approx \frac{\sqrt{\rho M}(q_2 - q_B) V''(q_2)}{1-\rho}\, y \approx \frac{\sqrt{2\rho M V''(q_2) \log(\rho^{-1})}}{1-\rho}\, y\,.$$

It remains to observe that Theorem 2.5.2 from [11] implies then that the ask price $p^a(t)$ satisfies

$$\mathbf{Pr}\left(p^a(t) \le q_B + y\right) \to \Phi(\tau)\,.$$

In other words, the price corresponding to a system of A_t such agents has mean q_B, shifted away from the well q_2 on a distance of order $(\log(\rho^{-1})/V''(q_2))^{1/2}$ and the variance of the price (ie, the volatility) diverges as $(MV''(q_2))^{-1}(\rho \log(\rho^{-1}))^{-1}$ in the limit of small $\rho = B_t/(M + B_t)$.

Finally, recalling the hydrodynamic description of B_t and the definition of T_c, we can also describe the time dependence of the mean $\mathbf{E}\rho$,

$$\mathbf{E}\rho = \mathbf{E}\rho_t \approx \frac{1 - \exp\{-(w_2 + w_1)(T_c - t)\}}{1 + w_1 \exp\{(w_2 + w_1)t\}/\bigl((w_2 + w_1)a_0 - w_1\bigr)}\,, \qquad t \le T_c.$$

3.2. **Interacting traders.** The simple model introduced in the previous section is rather artificial and simplistic. In reality one would expect that the behaviour of a trader is influenced by the information received from the market as well as external influences. Moreover, the opinion held by a trader with respect to the current price should somehow reflect some his intrinsic psychological characteristics. Finally, the exchange of shares occurring when a transaction takes place should have some visible effect on the time evolution. In the following we suggest some minimal features that should be incorporated in an interacting model to take into account some such effects. We will see that these features correspond to types of interactions that are not commonly considered in the theory of interacting particle systems.

- The derived process of the (ask and bid) price is the most easily accessible piece of information about the trading state of the market for any trader. It is natural that the updating rules should take the current value of this process into account. The simplest and natural modification is to introduce a bias towards the actual price $\bigl(p^a(t), p^b(t)\bigr)$ into the distribution of opinion change.
- Traders whose opinion is far from the current price are likely not to pay much attention to what is happening on the market. It is reasonable to assume that they update their opinion less frequently. This feature can be included by reducing the overall transition rates as a function of $p_i(t) - p(t)$.
- Finally, it is natural to assume that the traders performing a transaction, that is exchange of a share, will update their opinions according to some special rules reflecting the fact that someone buying or selling a share at a given price believes that she has struck a favourable deal, *i.e.* they attribute a higher value to the share then what they paid, respectively a lower one then what they got.

In the following we describe some concrete framework in which these features are implemented. We describe the construction of the process algorithmically.

Change of opinion: At any time step we first select at random a trader. We will allow this probability to depend on p_i, and we choose trader i with a probability proportional to $f(p_i(t) - p(t))$, where we define the "current price" via $p(t) = \big(p^a(t) + p^b(t)\big)/2$ and the function $f(x) \geq 0$ has its maximum at zero. The function f is responsible for the slow-down phenomenon away from $p(t)$. Once a trader has been selected, she changes her opinion from p_i to p_i' with probability proportional to $q(p_i, p_i')$ which in turn may depend on the entire state of the system. A possible choice for these functions is

$$f(x) = 1/(1 + |x|)^\alpha, \qquad \alpha > 0\,,$$

and $q(x, y; \mathbf{p})$ being, for any fixed $\mathbf{p}$, a kernel of a random walk. In typical cases, $q(x, y; \mathbf{p})$ depends on $\mathbf{p}$ through p^a and p^b only. Once p_i' is chosen, we check whether $p_i' < p^a$, if $n_i = 0$, resp. whether $p_i' > p^b$, if $n_i = +1$. If this is the case, we set $p_i(t + 1) = p_i'$, and $p_j(t + 1) = p_j(t)$ for all $j \neq i$, and continue to the next time step. Otherwise, we perform

Transaction: Assume first that $n_i(t) = 0$ and $p_i' \geq p^a(t)$. This means that the buyer i has decided to buy at the current asked price. Since by definition there is at least one seller who asks only the price $p^a(t)$, we select from all these one at random with equal probabilities. Call this trader j. Then we set

$$p_i(t + 1) = p^a(t) + g, \qquad p_j(t + 1) = p^a(t) - g$$

where $g > 0$ is a fixed or possibly random number. Similarly, if $n_i(t) = 1$ and $p_i' \leq p^b(t)$, the seller i sells to one of the buyers that offer the price $p^0(t)$, and we set

$$p_i(t + 1) = p^o(t) - g, \qquad p_j(t + 1) = p^o(t) + g$$

The final state in all cases represents a new stable trading state and the process continues. Note that g should be at least as large as to cover the transaction cost.

These additional features make the mathematical analysis of the model much more difficult, but they introduce some interesting effects that are somewhat similar to phenomena observed in real markets. Let us briefly comment on these.

In the ideal gas approximation in the absence of any confining potential all opinions would in the long run spread over all real numbers and the individual opinions would get arbitrarily far from each other. This is avoided by the mechanism of the *attraction to the current price.*

Slowing down the jump rate of the particles far away from the current price naturally introduces long time memory effects that lead to special features in the distribution of the price process. One of them is possible existence of resistances: if there is a large population of traders in a vicinity of some price p far from (say above) the current price $p(t)$, then this population tends to persist for a long time, unless the current price approaches this value. If that happens, *e.g.* due to the presence of an upward drift, one observes a slow-down of the upward movement of the price when it approaches this value. The market has a resistance against increase of the price through this value, reflected in multiple returns to essentially the same extremal values for a long period, see Fig 2.

A further effect of the slowing down is the tendency of the creation of "bubbles" in the presence of strong drifts. In this case one observes a fast motion of the price accompanied by a depletion of the population below this price. Effectively, a few

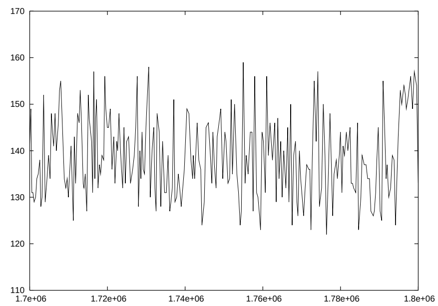

Figure 2. Trading in between resistances

(buying) traders move with the drift, while most are left behind. Such a situation can lead to a crash, if at some moment the drift is removed (due to external effects). Such a scenario was played out in a simulation that is shown in Figure 3.

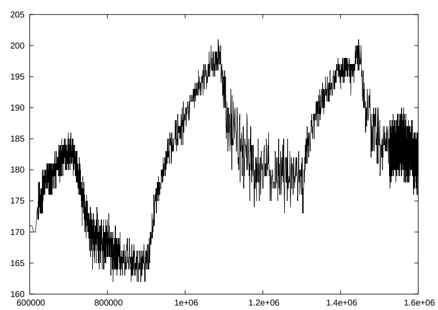

Figure 3. A sequence of "bubbles"

Note that after the crash there was a strong increase of volatility.

The effect of pushing the opinions from the current price after the *transaction* depletes the vicinity of the price and therefore increases the volatility. This effect goes in the opposite direction as the attraction to the price and the interplay of both effects can lead to a non-trivial quasi-equilibrium state. The effect of these mechanisms on the price fluctuation will be studied in forthcoming paper [3].

To illustrate the influence of different effects discussed above, we present in Fig. 4 and 5 simulation results of the crash-type scenario from Sect. 3.1 where three different parameters of the model were changed independently. All simulations are based on a discretized version of the diffusion model from Sect. 3.1 with the same potential function V having two local minima (one metastable and one stable). All simulations start with the same initial condition where all traders are located near the metastable minimum.

Figure 4 shows simulations without the feature of slowing down rates as a function of the distance to the current price: 1) corresponds to the free gas approximation; 2) shows the same scenario with an additional drift towards the current price; 3) and 4) are like 1) and 2) with a trading effect corresponding to g being uniformly distributed on the interval $[3, 10]$.

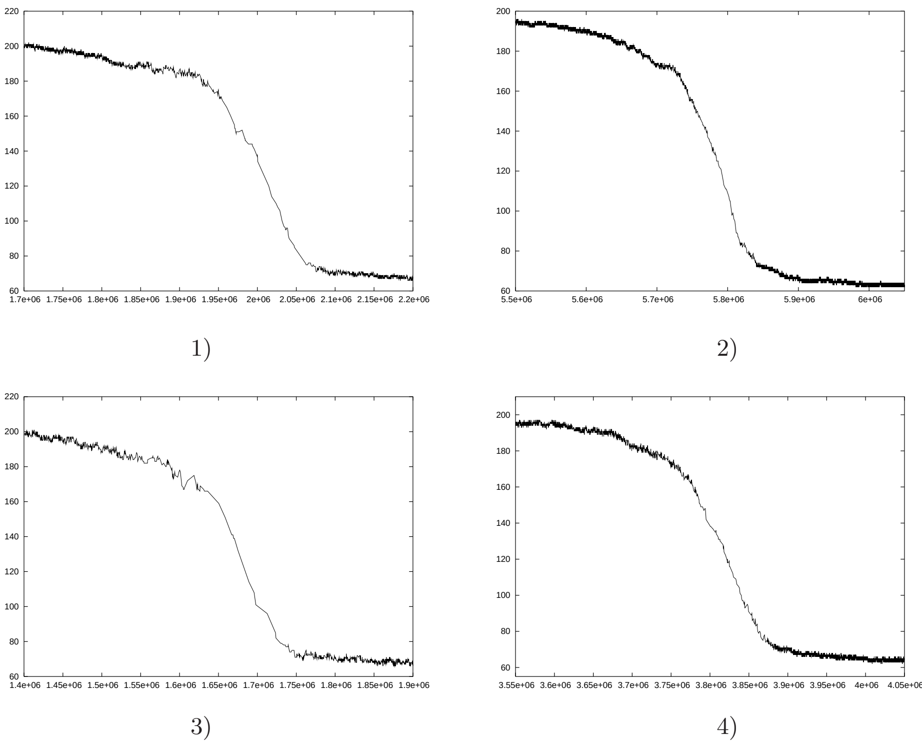

FIGURE 4. Crash scenario without slowdown

Figure 5 shows the same sequence of scenarios when the overall rate of updating of trader i behaves like $\big(p_i - p(t)\big)^{-1.5}$. Notice the increased volatility compared to the previous picture. The volatility increase before the crash is particularly marked.

4. Possible extensions

The basic model we describe above allows for numerous extensions to capture further important features.

4.1. **Traders of different type.** We have assumed all traders to behave according to the same stochastic rules. It is not difficult to modify this. First, rules can be different between buyers and sellers, generalizing the bias towards the price to some more complicated function. Moreover, we could introduce different species of traders that follow different rules (*e.g.* optimists *vs.* pessimists), and study the ensuing effects. Even more challenging, we could try to introduce "intelligent agents" that try to perform arbitrage on the price by estimating future price based on observation of the past price history. This goes beyond the original ideas of the model, but could be interesting when testing some basic principles of financial mathematics in a specific controllable model context.

4.2. **Coupling to external influences.** To capture the evolution of prices over longer time-scales, it will be indispensable to couple our model to external influences. These should reflect fundamental data on the particular stock considered

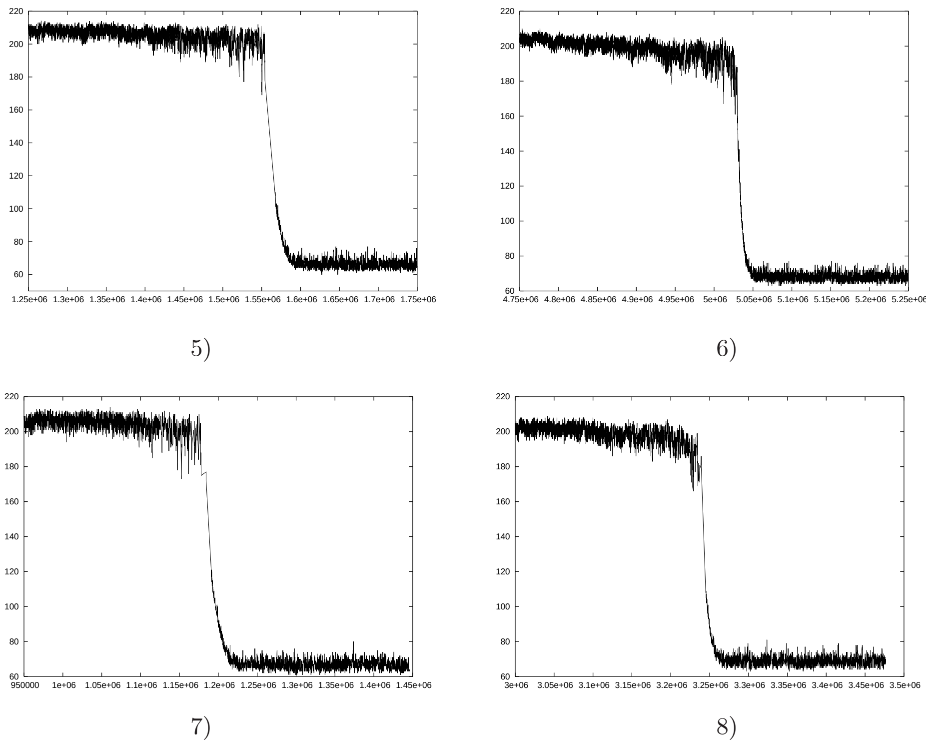

FIGURE 5. Crash scenario with slowdown

(such as dividend return, earnings, cash-flow), as well as global macro-economic data (interest rates, growth rates, etc.). Such effects are easily incorporated by making the transition rates $q(p_i, p_i')$ time dependent. *E.g.*, given the earnings at time t, one may compute a fictitious "fundamental value" (based *e.g.* on historic price-earnings ratios), and assume that there should be a certain tendency for market participants to adjust their subjective price towards this value. Changes in earnings (expectations) then induce a change in the transition probabilities. Similarly, other external effects exert their influence most naturally through the transition probabilities of our process. The key question of interest that our model is able to answer is how such external effects are reflected in the evolution of the price of our commodity. Addressing this question via analytical and/or numerical methods may in many respects be the most interesting and promising perspective that our models provide.

5. CONCLUSIONS

We have presented a class of Markov models that allow a realistic modelling of the price evolution of a commodity under trading. The basic model is a particle system like model for the evolution of a large number of traders whose state space is given by the collection of all opinions of all traders on the current value of the traded commodity. The price process is inferred from this state according to rules analogous to those used in real markets. In the simplest case of independent

traders, explicit computations are possible, and we have analysed a crash scenario in a bistable market in this context.

We discussed several basic mechanisms that we think should be taken into account when modelling financial markets. These include attraction to the price, rate dependence from the distance to the price, and repulsion from the price of traders having performed an interaction. These effects lead to interesting properties of the price process which are observed in similar form in reality.

We hope to have motivated that the interacting particle systems have a place in the modelling of financial and economic systems. For this to be fruitful, this requires to chose interactions that take the special features of these systems into account. Moreover, the questions that should be addressed are quite different from what is usually done in the theory of interacting particle systems. In particular the analysis of the price process leads to rather interesting problems regarding order statistics in the particle model. As we will discuss in a forthcoming article [3], these problems are closely related to the study of interfaces and phase boundaries in particle systems.

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